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Geometric Theory of Korovkin Sets

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INTRODUCTION

They key results and simplest examples for the present paper are furnished by the following theorems of P. P. Korovkin [12, [13]. They concern positive linear operators T on the space $C[a, b]$:

THEOREM I. *Let $\{T_n, n = 1, 2,...\}$ be a sequence of positive linear operators on $C[a, b]$, and for each of the functions $g_i(x) = x^i$, $i = 0, 1, 2$, let*

$$\lim_n T_n g_i(x) = g_i(x) \text{ uniformly on } [a, b].$$

Then, for all $f \in C[a, b]$,

$$\lim_n T_n(x) = f(x) \text{ uniformly on } [a, b].$$

In other words, the triple $1, x, x^2$ is a test set.

THEOREM II. *There is no test set for $C[a, b]$, consisting only of two functions.*

THEOREM III. *A triple f_0, f_1, f_2 is a test set exactly when it is a Cebysev system on $[a, b]$.*

For the proofs, see for example [13]. As predecessors of Korovkin,

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T. Popoviciu, and H. Bohman should be mentioned. Korovkin was the first, however, to consider arbitrary positive linear operators on $C[a, b]$.

An abstract formulation of the problem before us is as follows. Let E, F be two real or complex Banach spaces. Let $\mathscr{T}$ be a fixed subclass of the set of all continuous linear operators from E to F, and let P be a fixed operator of the class $\mathscr{T}$. A set $S \subset E$ is a Korovkin set with respect to $\mathscr{T}$, P, if for each sequence $T_n \in \mathscr{T}$, the relation $T_n g \to Pg$, $g \in S$ (in the norm of F) implies $T_n f \to Pf$, $f \in E$.

Our problem is to characterize Korovkin sets. In what follows, $E = F = C(X)$, and P is the identity operator I.

The present expository paper concerns itself with the geometric approach to Korokin's theorems and has its origin in the paper [23] of Šaškin. It is based on the lectures delivered by G.G. Lorentz at the Regional Conference at Riverside, California, in 1972. The mimeographed edition [17] of the lectures has been much improved in a course offered by H. Berens at Erlangen University in the Winter Semester 1973–74. In the present form the exposition is a product of both of us. We are grateful to Prof. Oved Shisha for his offer to publish it in the Journal of Approximation Theory.

1. Korovkin Sets in $C(X)$

1.1. *Basic Definitions*

1.1.1. Let X be a compact metric space, $C(X)$ be the space of all real-valued continuous functions f on X with the norm

$$\|f\| = \max\{|f(x)|: x \in X\},$$

and let $C^+(X)$ be the cone of positive functions in $C(X)$. The space $\mathscr{E}(C)$ of endomorphisms on $C(X)$ contains all positive linear transformations T that map $C^+(X)$ into itself: $Tf \geqslant 0$ whenever $f \geqslant 0$. For a positive transformation T one has $|Tf| \leqslant T|f|$; moreover, if 1_X denotes the constant function equal to 1 on X,

$$\|T\| = \|T1_X\|.$$

This follows at once from the inequalities $\|Tf\| \leqslant \|T(\|f\| 1_X)\| = \|f\| \, \|T1_X\|$. (Similar remarks apply to positive functionals on C.)

We denote by $\mathscr{T}^+$ the cone of positive transformations, by $\mathscr{T}_1$ the class of contractions T in $\mathscr{E}(C)$ characterized by $\|T\| \leqslant 1$, and by $\mathscr{T}_1^+$ that of positive contractions.

The purpose of this and the next section is to find characterizations of Korovkin sets $S \subset C(X)$ with respect to $\mathscr{T}$, which may be one of the three classes, $\mathscr{T}^+$, $\mathscr{T}_1$, and $\mathscr{T}_1^+$. The operator P will always be the identity I.

We give analytic descriptions of Korovkin sets, and later, geometric descriptions, based on notions of convex sets and extreme points.

Section 1 is based on the fundamental papers [22, 23] of Šaškin, which treated the case of the class $\mathscr{T}^+$ and of finite S. Later Šaškin [24] and Wulbert [28] indicated the possibility of a similar treatment of $\mathscr{T}_1$.

1.1.2. For a compact metric space X, we denote by $\mathscr{M}(X)$ the space of real regular Borel measures μ on X. The norm of $\mu \in \mathscr{M}$ is its total variation $\int | d\mu |$ on X. According to the Riesz representation theorem, $\mathscr{M}(X)$ is isometrically isomorphic to the dual space of $C(X)$. For the duality relation between $C(X)$ and $\mathscr{M}(X)$ we use both notations

$$\mu(f) \qquad \text{and} \qquad \int f \, d\mu.$$

We denote by $\mathscr{P}(X)$ the space of probability measures μ on X that are positive and satisfy $\mu(X) = 1$. Equivalently, they are characterized by the relation $\mu(1_X) = 1 = \| \mu \|$. Corresponding to the classes of transformations, we define: $\mathscr{L}^+$ as the cone of positive measures in $\mathscr{M}$, $\mathscr{L}_1$ as the unit ball in $\mathscr{M}$, and $\mathscr{L}_1^+ = \mathscr{L}^+ \cap \mathscr{L}_1$. $\mathscr{L}$ will stand for one of the three classes.

For given X and $\mathscr{T}$, we denote by $m(X, \mathscr{T})$ the smallest cardinal number of a Korovkin set $S \subset C(X)$ with respect to $\mathscr{T}$. One proves without difficulty:

PROPOSITION 1. *Let Φ be a homeomorphic mapping of X onto another compact metric space X'. If S is a Korovkin set for $C(X)$ and the class $\mathscr{T}$, then also $S' = \{g' = g \circ \Phi^{-1}: g \in S\}$ is a Korovkin set for $C(X')$ and the same class $\mathscr{T}$.*

Hence, the minimal order $m(X, \mathscr{T})$ is a topological invariant of X.

Later we will see that a Korovkin set S with respect to $\mathscr{T}$ in $C(X)$ satisfies the following conditions:

(i) S separates points of X: for each pair of points $x, x' \in X$, $x \neq x'$, there is a function $g \in S$ for which $g(x) \neq g(x')$.

(ii) S does not vanish on X: for each $x \in X$ there exists a $g \in S$ with the property $g(x) \neq 0$.

In view of this, it is reasonable to assume from the beginning that the set S satisfies (i), (ii). We call such S admissible. Condition (ii) is satisfied if S contains a strictly positive function $g_0(x) > 0$ for all $x \in X$.

1.2. *A Necessary and Sufficient Condition*

1.2.1. For a set $S \subset C(X)$, we shall denote by $G = \operatorname{lin} S$ its linear hull in $C(X)$, by $\bar{G} = \overline{\operatorname{lin}} \, S$ its closed linear hull, and by G^* the dual space of G.

It is convenient to carry out the proofs in Sections 1.2. and 1.3. at once for the three possible classes $\mathcal{T}$: $\mathcal{T}^+$, $\mathcal{T}_1$, and $\mathcal{T}_1^+$ with $\mathcal{L}$ denoting the *corresponding* class $\mathcal{L}^+$, $\mathcal{L}_1$, $\mathcal{L}_1^+$.

To each of these three $\mathcal{L}$ and each $x \in X$ we make correspond the set of functionals

$$\mathcal{L}_x(S) = \{\mu \in \mathcal{L}: \mu(g) = g(x), g \in S\}.$$

The evaluation functional ϵ_x, defined by $\epsilon_x(f) = f(x)$, obviously belongs to $\mathcal{L}_x$, but this set may contain further functionals. Let l_x be the restriction of ϵ_x to G. Then $\mathcal{L}_x$ consists exactly of all those functionals $\mu \in \mathcal{L}$ that are extensions of l_x.

If $1_X \in S$, then $\mathcal{L}_x = \mathcal{P}_x$, where

$$\mathcal{P}_x(S) = \{\mu \in \mathcal{P}: \mu(g) = g(x), g \in S\}.$$

1.2.2. We shall need some lemmas about the class $\mathcal{L}^+$.

LEMMA 1. *The linear hull G of $S \subset C(X)$ contains a strictly positive function exactly when the zero measure is the only measure in $\mathcal{L}^+$ that annihilates G.*

Proof. The condition is clearly necessary. To prove its sufficiency, assume that G does not contain a strictly positive function. The representation of functions from $G_1 = G + \mathbb{R}1_X$ in the form $g + \lambda 1_X$ is unique. It follows that the functional

$$l(g + \lambda 1_X) = \lambda$$

is well-defined on G_1. It annihilates G and is positive, since $g + \lambda 1_X \geqslant 0$ implies $\lambda \geqslant 0$. According to Krein's extension theorem [2, p. 12], l has a positive (hence continuous) linear extension $\mu \in \mathcal{L}^+$.

LEMMA 2. *Let S be a subset of $C(X)$ that satisfies for some $x \in X$ the condition*

$$\mathcal{L}_x^+(S) = \{\epsilon_x\}. \tag{1.2.1}$$

Then the linear hull G of S contains a strictly positive function.

Proof. Otherwise, according to Lemma 1, there exists a measure $\mu_0 \in \mathcal{L}^+$, $\mu_0 \neq 0$ that annihilates G. Then $\epsilon_x \neq \epsilon_x + \mu_0 \in \mathcal{L}_x^+(S)$, contradicting (1.2.1).

This statement is not necessarily true for $\mathcal{T}_1$ or $\mathcal{T}_1^+$ (see Example 9, Section 2.2).

LEMMA 3. *Let* (1.2.1) *be satisfied, and let $\{\mu_n: n = 1, 2,...\}$ be a sequence in $\mathcal{L}^+$ for which $\lim_n \mu_n(g) = g(x)$ for all $g \in S$. Then the norms $\|\mu_n\|$ are bounded.*

Proof. Let g_0 be a strictly positive function in G, and let $g_0(x) \geqslant c > 0$ for all $x \in X$. Then

$$\| \mu_n \| = \mu_n(1_X) \leqslant (1/c)\, \mu_n(g_0) \to (1/c)\, g_0(x).$$

PROPOSITION 2. *Let $S \subset C(X)$ and $x \in X$ be given. Then the condition*

$$\mathscr{L}_x(S) = \{\epsilon_x\} \tag{1.2.2}$$

is necessary and sufficient in order that for each sequence $\{\mu_n\colon n = 1, 2,...\}$ in $\mathscr{L}$, relations

$$\lim_n \mu_n(g) = g(x), \qquad g \in S \tag{1.2.3}$$

should imply

$$\lim_n \mu_n(f) = f(x), \qquad f \in C(X). \tag{1.2.4}$$

Proof. The necessity of the condition is obvious. If $\mu_0 \in \mathscr{L}_x$, $\mu_0 \neq \epsilon_x$, then the consideration of the sequence $\{\mu_n = \mu_0\colon n = 1, 2,...\}$ leads to a contradiction.

To prove the sufficiency, let (1.2.2) and (1.2.3) be satisfied. We establish (1.2.4) by using the weak* topology in $\mathscr{M}(X)$. Let $\{\mu_{n_k}\colon k = 1, 2,...\}$ be an arbitrary subsequence of $\{\mu_n\}$. The sequence of norms $\| \mu_{n_k} \|$ is bounded; for the case $\mathscr{L} = \mathscr{L}^+$ this follows from Lemma 3. The weak* compactness of balls in $\mathscr{M}(X)$ and the separability of $C(X)$ imply the existence of a subsequence $n_k' \to \infty$ and of an element $\mu_0 \in \mathscr{M}(X)$ for which $\mu'_{n_k} \to \mu_0$ in the weak* topology. Since $\mathscr{L}$ is weak* closed, $\mu_0 \in \mathscr{L}$. Now (1.2.3) implies $\mu_0(g) = g(x)$, $g \in S$, or $\mu_0 \in \mathscr{L}_x$. By (1.2.2) we have $\mu_0 = \epsilon_x$.

1.2.3. From Proposition 2 we can derive several corollaries.

THEOREM 1. *For a set $S \subset C(X)$, the conditions*

$$\mathscr{L}_x(S) = \{\epsilon_x\}, \qquad x \in X \tag{1.2.5}$$

are necessary and sufficient in order that for each sequence $\{T_n\colon n = 1, 2,...\}$ in $\mathscr{T}$, the relations

$$\lim_n T_n g(x) = g(x) \text{ pointwise on } X, \qquad g \in S \tag{1.2.6}$$

should imply

$$\lim_n T_n f(x) = f(x) \text{ pointwise on } X, \qquad f \in C(X). \tag{1.2.7}$$

Proof. The sufficiency of the condition is immediate. To prove the necessity, assume that there exists a point $x_0 \in X$ and a functional $\mu_0 \in \mathscr{L}_{x_0}(S)$,

$\mu_0 \neq \epsilon_{x_0}$. We prove more than is needed at present, namely, that there exists a sequence $\{T_n\}$ in $\mathscr{T}$ for which (1.2.6) holds *uniformly* on X, and that (1.2.7) fails.

Let $d(\ , \ )$ be the metric on X. For each $n = 1, 2,...$ we select a function $\phi_n \in C(X)$ with the properties $0 \leqslant \phi_n(x) \leqslant 1$, $\phi_n(x_0) = 1$, and $\phi_n(x) = 0$ for $d(x, x_0) \geqslant 1/n$. We define

$$T_n f(x) = \mu_0(f) \cdot \phi_n(x) + f(x) \cdot [1 - \phi_n(x)], \qquad n = 1, 2,... . \tag{1.2.8}$$

This sequence $\{T_n\}$ belongs to $\mathscr{T}$, and for each $g \in S$

$$\lim_n T_n g(x) = g(x) \qquad \text{uniformly on } X.$$

On the other hand, there exists a function $f_0 \in C(X)$, for which $\mu_0(f_0) \neq f_0(x_0)$. For this function we have

$$T_n f_0(x_0) = \mu_0(f_0) \neq f_0(x_0), \qquad n = 1, 2,... .$$

1.2.4. According to a classical result, a sequence $\{f_n : n = 1, 2,...\}$ in $C(X)$ converges weakly to an element $f_0 \in C(X)$ if and only if the sequence of norms $\{\| f_n \|\}$ is bounded, and $\lim_n f_n(x) = f_0(x)$ pointwise on X (see [9, p. 265]). For $\mathscr{T}_1$ and $\mathscr{T}_1^+$ this yields the following theorem.

THEOREM 2. *The statement of Theorem* 1 *remains true with pointwise convergence replaced by weak convergence in* $C(X)$.

We leave the proof for the class $\mathscr{T}^+$ to the reader.

1.2.5. We now give the main result of this section, which characterizes the Korovkin sets of $C(X)$ for uniform convergence.

THEOREM 3. *A necessary and sufficient condition for a set* S *in* $C(X)$ *to be a Korovkin set with respect to* $\mathscr{T}$ *is that*

$$\mathscr{L}_x(S) = \{\epsilon_x\} \qquad \textit{for all } x \in X. \tag{1.2.9}$$

Proof. The necessity follows from the proof of Theorem 1. To show the sufficiency, assume that there is a sequence $\{T_n : n = 1, 2,...\}$ in $\mathscr{T}$ which enjoys the properties $T_n g \to g$ for $g \in S$ and $T_n f_0 \not\to f_0$ for some $f_0 \in C(X)$. Then there exists an $\epsilon > 0$, a sequence $\{n_k : k = 1, 2,...\}$ and a sequence of points $\{x_k : k = 1, 2,...\}$ in X for which

$$\epsilon \leqslant | T_{n_k} f_0(x_k) - f_0(x_k)|.$$

Since X is compact, we can assume that x_k converges, say to $x_0 \in X$. We define the sequence $\{\mu_k: k = 1, 2, \ldots\}$ of functionals by means of the formula

$$\mu_k(f) = T_{n_k} f(x_k).$$

The functionals μ_k belong to $\mathscr{L}$ and satisfy

$$\lim_k \mu_k(g) = g(x_0), \qquad g \in S.$$

According to Proposition 2, $\mu_k \to \epsilon_{x_0}$ weak* for $k \to \infty$. In particular, $\mu_k(f_0) \to f_0(x_0)$ for $k \to \infty$, and this is a contradiction.

From Theorem 3 one easily derives conditions (i) and (ii) of 1.1.2: If (i) is violated, one should compare ϵ_x, $\epsilon_{x'}$; if (ii) is violated, one compares ϵ_x and 0.

COROLLARY 1. *A subset S of $C(X)$, which contains the function 1_X, is a Korovkin set in $C(X)$ with respect to $\mathscr{T}$ if and only if*

$$\mathscr{P}_x(S) = \{\epsilon_x\} \qquad \textit{for all } x \in X. \tag{1.2.10}$$

1.2.6. We show that for the case of $\mathscr{T}^+$, the consideration of Korovkin sets can be reduced to sets containing the function 1_X.

PROPOSITION 3. *If S is a Korovkin set with respect to $\mathscr{T}^+$ in $C(X)$, then the same holds for the set $S' = \{g' = g_0 g: g \in S\}$, where g_0 is an arbitrary strictly positive function in $C(X)$.*

Proof. We select a point $x \in X$ and have to show that $\mathscr{L}_x^+(S') = \{\epsilon_x\}$. Let $\mu_0 \in \mathscr{L}_x^+(S')$. Then

$$\mu_0(g_0 g) = g_0(x)\, g(x), \qquad g \in S.$$

The measure γ, defined by

$$\gamma(f) = \frac{1}{g_0(x)}\, \mu_0(g_0 f)$$

is positive and belongs to $\mathscr{L}_x^+(S)$. But according to Theorem 3, the last set consists only of the functional ϵ_x. Thus, $\gamma = \epsilon_x$ and, consequently, $\mu_0 = \epsilon_x$.

PROPOSITION 4. *Let S be a Korovkin set of order m with respect to $\mathscr{T}^+$ in $C(X)$, then there exists another such set S which contains 1_X and has order not exceeding m.*

Proof. According to Theorem 3 and Lemma 2, the set $G = \operatorname{lin} S$ contains a strictly positive function g_0. By Proposition 2, the set $S' = \{g' = g/g_0$:

$g \in S\}$ is also a Korovkin set. The dimension of $G' = \operatorname{lin} S'$ is the same as that of G, hence at most $m+1$, and $1_X \in G'$. We can select for G' a basis S_0 of at most $m+1$ functions, which contains 1_X. This will be the required Korovkin set.

Our purpose now will be to replace the analytic conditions (1.2.9) and (1.2.10) (in Theorem 3 and Corollary 1) by simpler conditions of geometric character. This will be especially apparent in Section 2, where our conditions express simple properties of convex sets in the Euclidean space $\mathbb{R}^{m+1}$. For this purpose we need the basic notions and theorems about convex sets in vector spaces, which we assume to be known. See, for example, [2, 9, 14, 20].

1.3. *Sets Containing the Function 1_X, Choquet Boundary, and Peak Points*

1.3.1. In this section, S_0 will be a subset of $C(X)$ that separates points and contains the function 1_X; hence, it is admissible. We discuss conditions under which S_0 is a Korovkin system. As before, let $G = \operatorname{lin} S_0$, $\bar{G} = \overline{\operatorname{lin}} S_0$, and let G^* be the dual space of G. For $x \in X$, $l_x \in G^*$ is the functional given by $l_x(g) = g(x)$, $g \in G$. Of basic importance for us is the map

$$\Phi: x \to l_x, \qquad x \in X \tag{1.3.1}$$

which sends points x of X into functionals $l_x \in G^*$. By assumption, this mapping is one-to-one. We denote by X^* the image of X under Φ in G^*. If G^* is equipped with the weak* topology, Φ is continuous, hence X^* is weak* compact as the image of the compact set X. We see that Φ is a homeomorphism. Let $K = \overline{\operatorname{co}}^* X^*$ be the weak* closed convex hull of X^* in G^*. Also K is weak* compact, moreover we have

LEMMA 4. $K = \{l \in G^*: l(1_X) = 1 = \|l\|\}$.

Proof. Let K_0 be the right-hand side of the equation, then obviously $K \subset K_0$. Assume that there exists an $l_0 \in K_0 \backslash K$, then l_0 and K can be separated by an element $g \in G$, so that

$$\sup\{g(x): x \in X\} \leqslant \sup\{l(g): l \in K\} < l_0(g). \tag{1.3.2}$$

This inequality is not destroyed if a constant is added to g. Selecting this constant properly, we will have $\|g\| = \sup\{g(x): x \in X\}$, and then (1.3.2) is a contradiction, since $l_0(g) \leqslant \|g\|$.

According to the theorem of Krein–Mil'man, the set of extremal points of K is not empty, and by Milman's addition to the theorem, the set of the extremal points is contained in X^*. (Mil'mans addition states that the extreme points of $\overline{\text{co}}\, M$ belong to the closure of M.) Of importance for us will be

PROPOSITION 5 [2]. *Let $x \in X$, then the functional l_x is an extremal point of K if and only if*

$$\mathscr{P}_x(S_0) = \{\epsilon_x\}. \tag{1.3.3}$$

Proof. First assume that (1.3.3) is violated. Then the functional l_x has an extension $\mu \in \mathscr{P}_x$ that is not identical with ϵ_x. Because the measure μ is regular, there exists a compact set $D \subset X \backslash \{x\}$, for which $\mu(D) > 0$. Since D is compact, there exists a point $x_0 \in X$ and a decreasing sequence $\{D_k: k = 1, 2,...\}$ of compact sets for which

$$\bigcap_{k=1}^{\infty} D_k = \{x_0\} \qquad \text{and} \qquad \lambda_k = \mu(D_k) > 0, \qquad k = 1, 2, \ldots .$$

If we had $\lambda_k = 1$ for all k, then it would follow that $\mu = \epsilon_{x_0}$, hence $l_x = l_{x_0}$, which contradicts 1.1.2(i). Hence, $\lambda_k < 1$ for all large k.

We define the sequence of probability measures

$$\mu_k(B) = \lambda_k^{-1} \cdot \mu(B \cap D_k), \qquad k = 1, 2, \ldots, \qquad B \in \mathscr{B},$$

where $\mathscr{B}$ are the Borel subsets of X. This sequence converges weak* in $\mathscr{M}(X)$ to ϵ_{x_0}. The restriction l_k of μ_k to G belongs to K and satisfies

$$l_k \to l_{x_0} \neq l_x \qquad \text{for} \quad k \to \infty,$$

weak* in G^*. We fix a k for which $l_k \neq l_x$, $\lambda_k < 1$, and define

$$\mu_k'(B) = (1 - \lambda_k)^{-1} \mu(B \cap D_k') \qquad k = 1, 2, \ldots, \quad B \in \mathscr{B},$$

where D_k' is the complement of D_k in X. The relation

$$\mu(B) = \mu(B \cap D_k) + \mu(B \cap D_k') = \lambda_k \mu_k(B) + (1 - \lambda_k)\, \mu_k'(B)$$

yields

$$l_x = \lambda_k l_k + (1 - \lambda_k)\, l_k',$$

with l_k, $l_k' \in K$, $0 < \lambda_k < 1$, and $l_k \neq l_x$. This means that l_x is not an extremal point of K.

Conversely, let l_x not be an extremal point of K, then $l_x = \lambda l_1 + (1 - \lambda)\, l_2$ for $0 < \lambda < 1$ and some $l_1, l_2 \in K$, $l_1 \neq l_x$. Let $\mu_1, \mu_2 \in \mathscr{P}$ be any two extensions of l_1, l_2, respectively, and let $\mu = \lambda \mu_1 + (1 - \lambda)\, \mu_2$. Since $\mu_1 \neq \epsilon_x$, we have $\mu_1(\{x\}) < 1$, hence also $\mu(\{x\}) < 1$. Thus we obtained a measure μ in $\mathscr{P}_x(S_0)$ that is different from ϵ_x.

Let S_0 be as above, and let $G = \operatorname{lin} S_0$. The set of points $x \in X$ for which l_x is an extremal point of K, is called the *Choquet boundary*, $\partial_{\mathrm{ch}} G$, of G.

With this terminology we obtain from Corollary 1 of Theorem 3 and Proposition 5 the following theorem (Šaškin for $\mathscr{T} = \mathscr{T}^+$ [23], Wulbert [28], and Šaškin [24] for $\mathscr{T} = \mathscr{T}_1$):

THEOREM 4. *Let S_0 be a subset of $C(X)$ that separates points of X and contains the function 1_X. Then S_0 is a Korovkin set with respect to $\mathscr{T}$ in $C(X)$ exactly when*

$$\partial_{\mathrm{ch}} G = X, \qquad G = \operatorname{lin} S_0 . \tag{1.3.4}$$

1.3.2. In this subsection we discuss the notion and properties of peak points, important for the study of the Choquet boundary. A more detailed exposition of Choquet theory can be found in the books of Alfsen [1], Bauer [2], and Phelps [20].

To justify the name of boundary for $\partial_{\mathrm{ch}} G$, we make the following remarks. Let G be a closed subspace of $C(X)$, separating points and containing the function 1_X. A subset $B \subset X$ is called a *boundary* for G, if for each $g \in G$ there is a point $x \in B$ for which $|g(x)| = \|g\|$. (Thus, the name comes from the theory of analytic or harmonic functions). One can prove: *The set $\partial_{\mathrm{ch}} G$ in X is a boundary for G.* The smallest closed boundary for G, if it exists, is called the Šilov boundary for G. It is not difficult to show that in the above situation, the Šilov boundary exists and is identical with the closure of the Choquet boundary for G.

Later, we will need generalizations of the Choquet boundary based on slightly different ideas. A point $x_0 \in X$ is a *peak-point* of G (Bishop) if there exists a $g_1 \in G$ for which $g_1(x_0) = \|g_1\|$, $|g_1(x)| < \|g_1\|$, $x \neq x_0$. This is not the only possible definition. In Section 1.5 we learn two further ones. Under the assumption that $1_X \in G$, they are equivalent to the present one.

For the set $p(G)$ of peak-points we have the obvious inclusion

$$p(G) \subset \partial_{\mathrm{ch}} G. \tag{1.3.5}$$

A subset S of $C(X)$ for which $p(G) = X$, is called a *strict Korovkin set for $\mathscr{T}$ in $C(X)$*. For example, $S = \{1, x, x^2\}$ has the property. Thus, Korovkin's theorem holds not only for positive transformations, but also for contractions. In Section 2, we shall see an example of a Korovkin set that is not strict.

In a real Banach space E, we consider a point of the unit sphere of E, $\|f\| = 1$. The point f is *smooth* if there exists precisely one hyperplane passing through f and supporting the unit ball. In other words, f is smooth if there exists a unique element f^* of the unit sphere of the dual space for which $f^*(f) = 1$. For example, consider the unit sphere of the space $\bar{G} = \overline{\text{lin}}\, S$, where S is a Korovkin subset of $C(X)$. An element $g \in \bar{G}, \| g \| = 1$ is smooth if and only if $| g(x)|$ attains its maximum on X exactly once. For if $g(x_1) = \pm 1$, $g(x_2) = \pm 1$, $x_1 \neq x_2$, then there are two different functionals g^* with the required property, namely, the restrictions of $\pm \epsilon_{x_1}$ and $\pm \epsilon_{x_2}$ to $\bar{G}$ with properly chosen signs. In the following we shall need [20]:

PROPOSITION 6 (Mazur). *For a separable Banach space, the set of all smooth points of the unit sphere is a dense G_δ-set.*

By means of Mazur's theorem, Mil'man [19] proves a counterpart of (1.3.5):

PROPOSITION 7. *If $1_X \in G$ and if S separates points, then*

$$\partial_{\text{ch}} G \subset \overline{p(G)}.$$

We give proofs of generalizations of this in Section 1.5, Theorem 9, and after Theorem 8.

1.4. *Geometric Characterization of Korovkin Sets*

1.4.1. The results of Section 1.3 are simple and straightforward. Since $1_X \in S_0$, the same conditions and constructions work for the three cases $\mathscr{T}^+$, $\mathscr{T}_1$, $\mathscr{T}_1^+$. For general sets S, however, the conditions are slightly different in the three cases.

Let S be an admissible subset of $C(X)$, let G, G^*, the map Φ, $X^* \subset G^*$ be as before. The three cases are characterized by the existence of an interval $I \subset \mathbb{R}$ (namely $I^+ = [0, +\infty)$, $I_1 = [-1, +1]$, $I_1^+ = [0, 1]$) with the property that if $f(x) \in I$, $x \in X$, then $\mu(f) \in I$ for $\mu \in \mathscr{L}$. Let $I \times X^*$ denote all functionals of the form λl, $\lambda \in I$, $l \in X^*$; we put

$$K = \overline{\text{co}}^*(I \times X^*). \tag{1.4.1}$$

LEMMA 5. (i) *The set K of* (1.4.1) *consists of all functionals $l \in G^*$ that satisfy $l(g) \in I$ for each $g \in G$ with the property $g(x) \in I$, $x \in X$.*

(ii) *Equivalently, K consists of the restriction to G of all functionals $\mu \in \mathscr{L}(G)$.*

Proof. That (i) is equivalent to (ii) follows from different forms of Hahn–Banach theorem.

To prove (i), we denote the set of $l \in G^*$, described in (i), by K_0. Clearly, K_0 is weak* closed and convex, and contains K. Assume that there exists an $l_0 \in K_0 \backslash K$. Then l_0 can be separated from K by means of a hyperplane in G^*, given by some $g_0 \in G$. This means that

$$\sup\{l(g_0): l \in K\} < l_0(g_0). \tag{1.4.2}$$

This leads to a contradiction: In case $\mathscr{T}^+$, (1.4.2) means that $\lambda g_0(x) < l_0(g_0)$ for all $\lambda \geqslant 0$ and all $x \in X$. Consequently, $g_0(x) \leqslant 0$ for all x, and $l_0(g_0) \leqslant 0$. In case $\mathscr{T}_1$, one gets $|g_0(x)| < l_0(g_0)$ for all x, or $\| g_0 \| < l_0(g_0) \leqslant \| g_0 \|$. Similarly, for $\mathscr{T}_1^+$ we get $g_0^+(x) < l_0(g_0) \leqslant l_0(g_0^+) \leqslant \| g_0^+ \|$ for all $x \in X$.

In case $\mathscr{T}^+$, the set $K = K^+$ is the closed convex cone in G^* with the vertex in the origin, spanned by the set X^*. The set $K = K_1$ is the closed unit ball of G^*; equivalently, $K_1 = \overline{\text{co}}^*(X^* \cup (-X^*))$. Finally, $K_1^+ = K^+ \cap K_1 = \overline{\text{co}}^*(X^* \cup \{0\})$.

In case $\mathscr{T}^+$ we note some simple necessary conditions that must be satisfied for a Korovkin set S. Let ρ_x be the ray in K^+ generated by $x \in X$: $\rho_x = \{\lambda l_x: \lambda \geqslant 0\}$. From condition (1.2.9) of Theorem 3 it follows, as a necessary condition for a Korovkin set S:

$$\rho_x \neq \rho_{x'} \qquad \text{if} \quad x \neq x', \quad x, x' \in X. \tag{1.4.3}$$

Another simple necessary condition is given by

LEMMA 6. *The cone K^+ is acute*: $K^+ \cap (-K^+) = \{0\}$ *if and only if G contains a strictly positive function g_0.*

Proof. It is clear that the condition is sufficient. To prove its necessity, let J be the intersection of K^+ with the unit sphere in G^*, and let K be its weak* closed convex hull; K is weak* compact. The assumption is equivalent to the statement that 0 is an extreme point of K^+. Then 0 does not belong to K. For otherwise it would be an extreme point of K, which is impossible, since by Mil'man's theorem all extreme points of K belong to J. Then we can separate 0 and K by a hyperplane in G^* with an equation $l(g_0) = 1$, $l \in G^*$. The function g_0 is obviously strictly positive.

A ray ρ of a cone K with vertex in 0 is called extreme, if whenever $l \in \rho$ and $l = \lambda l_1 + (1 - \lambda) l_2$, $l_1, l_2 \in K$, then $l_1, l_2 \in \rho$. We have:

PROPOSITION 8. *Assume that the cone K^+ is acute and that* (1.4.3) *is satisfied. Then a ray ρ_x is an extreme ray of K^+ exactly when*

$$\mathscr{L}_x^+(S) = \{\epsilon_x\}. \tag{1.4.4}$$

Proof. The proof is similar to that of Proposition 5, and we shall omit the details. First let (1.4.4) be violated. Then there exists a measure $\mu \in \mathscr{L}_x^+$, $\mu \neq \epsilon_x$. If the mass of μ is concentrated at x, then $\mu = \alpha\epsilon_x$, and applying this to g_0, we obtain $\alpha = 1$. Hence, there exists a point $x_0 \neq x$ and a decreasing sequence $\{D_k: k = 1, 2, \ldots\}$ of compact subsets, not containing x, whose intersection is $\{x_0\}$, with the property that $\lambda_k = \mu(D_k) > 0$ for all k. If for all k, λ_k is the constant $\lambda_0 = \mu(X) > 0$, then $\mu = \lambda_0 \epsilon_{x_0}$. Restriction to G would give $l_x = \lambda_0 l_{x_0}$, contrary to the assumption (1.4.3). Hence, $\lambda_k < \lambda_0$ for all large k. We define the measures in $\mathscr{L}^+$,

$$\mu_k(B) = \lambda_k^{-1}\mu(B \cap D_k), \qquad \mu_k' = (\lambda_0 - \lambda_k)^{-1}\,\mu(B \cap D_k')$$

for any Borel measurable set B, and denote the restriction of the corresponding functionals by l_k, l_k'. Since $\mu_k \to \epsilon_{x_0}$ weak* in $\mathscr{M}(X)$, and consequently $l_k \to l_{x_0} \notin \rho_x$ weak* in G^*, we have $l_k \notin \rho_x$ for all large k. The relation between μ_k, μ_k', μ yields

$$l_x = \frac{\lambda_k}{\lambda_0}\lambda_0 l_k + \left(1 - \frac{\lambda_k}{\lambda_0}\right)\lambda_k l_k', \qquad 0 < \frac{\lambda_k}{\lambda_0} < 1.$$

This means that l_x is an interior point of the interval $[\lambda_0 l_k, \lambda_0 l_k']$ with end points in K^+, which is not contained in ρ_x. Thus, ρ_x is not an extremal ray of K^+.

From Theorem 3 and the necessity of Conditions (i) and (ii) below, we obtain

Theorem 5. *An admissible set S in $C(X)$ is a Korovkin set with respect to $\mathscr{T}^+$ exactly when the following conditions are satisfied:*

(B^+)
(i) $\rho_x \neq \rho_x'$ *for* $x \neq x'$, $x, x' \in X$
(ii) $K^+ \cap (-K^+) = \{0\}$
(iii) *for each* $x \in X$, ρ_x *is an extremal ray of* K^+.

We turn our attention to the case $\mathscr{T}_1$. A necessary condition for $S \subset C(X)$ to be a Korovkin set is

$$X^* \cap (-X^*) = \varnothing. \tag{1.4.5}$$

For otherwise $l_x = -l_{x'}$ for some $x \neq x'$, and then $\mathscr{L}_{1,x}$ contains the element $-\epsilon_{x'}$, different from ϵ_x.

The transformation $l \to -l$ is an isomorphism of K_1 onto itself; it maps X^* onto $-X^*$. It follows that l_x and $-l_x$ are at the same time extreme points of K_1 or they are not.

Proposition 9. *Let* (1.4.5) *be satisfied. For a point* $x \in X$,

$$\mathscr{L}_{1,x}(S) = \{\epsilon_x\} \tag{1.4.6}$$

holds if and only if l_x is an extreme point of K_1.

Proof (similar to proofs of Propositions 5 and 8). Assume that there exists a $\mu \in \mathscr{L}_{1,x}$, $\mu \neq \epsilon_x$. We have $\|\mu\| = 1$, $|\mu|(X) = 1$. If the total mass of μ is concentrated at x, then $|\mu| = \epsilon_x$; hence, $\mu = \epsilon_x$ (even if all $g \in G$ vanish at x). Hence, there exists a compact set $D_0 \subset X\backslash\{x\}$ with $\mu(D_0) \neq 0$. By Hahn's decomposition theorem, we can find a compact subset $D \subset D_0$, $\mu(D) \neq 0$, on which μ is positive (or negative). Using (1.4.5) we can prove (in a way similar to the proof of Proposition 5) that l_x is a nontrivial convex combination of points of K_1. The proof of the inverse is easy.

From this and Theorem 3 we obtain:

THEOREM 6. *An admissible set S in $C(X)$ is a Korovkin set with respect to $\mathscr{T}_1$ if and only if*

(B_1) *it satisfies* (1.4.5) *and in addition all points of X^* are extreme points of K_1.*

By Mil'man's theorem, the last condition is equivalent to

$$\operatorname{ext} K_1 = X^* \cup (-X^*). \tag{1.4.7}$$

We formulate the result for $\mathscr{T}_1^+$ without proof.

PROPOSITION 10. *An element $l_x \in X^*$ is an extreme point of K_1^+ exactly when*

$$\mathscr{L}_{1,x}^+(S) = \{\epsilon_x\}.$$

THEOREM 7. *An admissible subset S of $C(X)$ is a Korovkin subset for $\mathscr{T}_1^+$ exactly when each point of X^* is an extreme point of K_1^+, i.e.,*

(B_1^+) $$\operatorname{ext} K_1^+ \supset X^*.$$

1.5. *Korovkin Sets and Peak Points*

In the classical case, when $1_X \in S$ and S separates points, there are two equivalent definitions of the Choquet boundary by means of extremal points and by means of the condition $\mathscr{L}_x = \{\epsilon_x\}$. In the Sections 1.4 and 1.5, we do not assume that $1_X \in S$, and in [4], S need not even be admissible in the sense of Section 1.1. In such situations, relevant for us is the second definition:

Let S be a subset of $C(X)$ and let $G = \operatorname{lin} S$. For each of the three possible classes $\mathscr{T}$, the set of all points $x \in X$ that satisfy $\mathscr{L}_x(S) = \{\epsilon_x\}$, will be called the *generalized Choquet boundary* ∂G of G, or simply boundary of G with respect to $\mathscr{T}$.

The theorems of Section 1.4 have obvious formulations in terms of ∂G. In the case of $\mathscr{T}^+$, if h is a strictly positive function in $C(X)$ and $S' = \{g' : g' = g/h, g \in S\}$, $G' = \operatorname{lin} S'$, we have

$$\partial_{\text{ch}} G' = \partial_{\text{ch}} G. \tag{1.5.1}$$

Indeed, if $\mathscr{L}_{x_0}^+(S) = \{\epsilon_{x_0}\}$, we can show that $\mathscr{L}_{x_0}^+(S') = \{\epsilon_{x_0}\}$. Let $\mu \in \mathscr{L}_{x_0}^+(S')$, then $\mu \geqslant 0$ and $\mu(g') = g'(x_0)$, $g' \in G'$. Then $\mu_1(f) = \mu(f/h)\, h(x_0)$ belongs to $\mathscr{L}^+$ and satisfies $\mu_1(g) = g(x_0)$, $g \in G$; hence, $\mu_1 = \epsilon_{x_0}$ and $\mu = \epsilon_{x_0}$.

If we do not assume that $1_X \in S$, the definition of a peak point, given in Section 1.3.2, proves to be suitable only if $\mathscr{T} = \mathscr{T}_1$. We define a point $x_0 \in X$ to be a *peak point for G and the class* $\mathscr{T}$ if there is a function $g_0 \in G$ with the following properties:

(P^+) In case of $\mathscr{T}^+$, g_0 must satisfy $g_0(x_0) = 0$, $g_0(x) > 0$, $x \neq x_0$ (here, x_0 is a *zero minimum* point).

(P_1) For $\mathscr{T}_1$, the condition is $|g_0(x_0)| = 1$ and $|g_0(x)| < 1$, $x \neq x_0$ (a *maximum modulus* point).

($P_1{}^+$) For $\mathscr{T}_1{}^+$, we require that $g_0(x_0) > 0$, $g_0(x) < g_0(x_0)$ for $x \neq x_0$ (a *positive maximum* point).

We leave it to the reader to prove:

PROPOSITION 11. *If x_0 is a peak point for $\mathscr{T}$, then $\mathscr{L}_{x_0}(S) = \{\epsilon_{x_0}\}$. The three definitions of peak points coincide if $1_X \in G$.*

As a corollary we obtain: If the set of peak points of G satisfies

$$p(G) = X, \tag{1.5.2}$$

then S is a Korovkin system for $\mathscr{T}$. Korovkin sets of this kind are called *strict Korovkin sets.*

Of importance is also the weaker notion of quasi-peak-points, which we formulate first for the case $\mathscr{T}^+$.

A point $x_0 \in X$ is a *quasi-peak-point* of G if for each $\epsilon > 0$ and each neighborhood U of x_0 there exists a function $g_0 \in G^+$ for which $g_0(x) \geqslant 1$ for $x \in X \backslash U$ and $g_0(x_0) < \epsilon$.

For the set $q^+(G)$ of quasi-peak-points we have the inclusions

$$p^+(G) \subset q^+(G) \subset \partial^+ G. \tag{1.5.3}$$

Only the last inclusion needs a proof. If $x_0 \in q^+(G)$, $\mu \in \mathscr{L}_{x_0}^+$, we show that $\mu = \epsilon_{x_0}$. Applying μ to the function g_0 of the definition, we have $\mu(g_0) = g_0(x_0) < \epsilon$; hence, measure μ of the set $X \backslash U$ is $< \epsilon$. Since this is true for each pair U, ϵ the whole mass of μ is concentrated at x_0.

THEOREM 8. *If S is a subset of $C(X)$ and if G contains a strictly positive function g_0, then for the class $\mathscr{T}^+$,*

$$\partial^+ G = q^+(G) \subset \overline{p^+(G)}. \tag{1.5.4}$$

Proof. Inclusion $q^+(G) \subset \partial^+ G$ will be proven in a forthcoming paper [18]. Together with (1.5.3), this gives the first part of (1.5.4). To prove the second relation, we define $S' = \{g' = g/g_0 : g \in S\}$, then $1_X \in S'$. Clearly $p^+(G') = p^+(G)$. Using (1.5.1) and Proposition 7, we obtain the result.

Remark. Using the first relation (1.5.4), we can prove Proposition 7 without the assumption that S separates points. In fact, assume $1_X \in G$ and let $x_0 \in \partial^+ G$; then for given $0 < \epsilon < 1$, and a neighborhood U of x_0, there exists a function $g_0 \in G$ from the definition of a quasi-peak-point. Substracting a constant if necessary, we may assume that g_0 vanishes in a point of U. Then $1 - g_0/\| g_0 \|$ is positive and belongs to the unit sphere of G. We approximate this function by a smooth function $g_1 \in \bar{G}$; then g_1 will attain a strict maximum on U. Therefore $1 - g_1$ will have a minimum equal to zero in exactly one point $x_1 \in U$.

THEOREM 9. *If S is a subset of $C(X)$ that separates points of X, then*

$$\partial_1 G \subset \overline{p_1(G)}. \tag{1.5.5}$$

Proof (patterned after [19]). It is sufficient to show that the image of $p_1(G)$ under the homeomorphism Φ is dense in the set of extreme points of K_1, that is, in the image of ∂G. We select an extreme point l_{x_0} and show that it belongs to the weak* closure of the set $Y = \{l_x : x \in p_1(G)\}$. Clearly,

$$K_1 \supset \overline{\text{co}}\{l_x : x \in p_1(G)\}. \tag{1.5.6}$$

If we can show that l_{x_0} belongs to K', the right-hand side of (1.5.6), then l_{x_0} would be an extreme point of K', and the theorem of Mil'man would give that l_{x_0} belongs to the closure of Y, hence x_0 to the closure of $p_1(G)$. Assume that $l_{x_0} \notin K'$; then l_{x_0} can be separated from K' by a hyperplane given by some $g_0 \in G$. This means that

$$\alpha = \sup\{\lambda l_x(g_0) : |\lambda| \leqslant 1, x \in p_1(G)\} < l_{x_0}(g_0),$$

or that

$$|g_0(x)| \leqslant \alpha < g_0(x_0), \qquad x \in p_1(G). \tag{1.5.7}$$

We assume $\| g_0 \| = 1$ and we approximate g_0 by a smooth function g_1 of the unit sphere in $\bar{G}$, which attains unique maximum modulus at some point $x_1 \in X$. Then $|g(x)| < |g_1(x_1)|$ for all $x \in p_1(G)$, which is a contradiction, since $x_1 \in p_1(G)$.

Sometimes the following is useful in determining ∂G.

PROPOSITION 13. *Let $g_k \in G$, $k = 1, 2, \ldots$, and let A_k be the set where g_k vanishes. Assume that g_n is positive on the set $A_1 \cap \cdots \cap A_{n-1}$ (for $n = 1$, this means $g_1 \geqslant 0$). If $\bigcap_k A_k = \{x_0\}$, then $x_0 \in \partial G$.*

Proof. By induction one shows that if $\mu \in \mathscr{L}_{x_0}^+(G)$, then μ is concentrated on A_n.

THEOREM 10. *Each Čebyšev system S (with more than three functions) on $X = [a, b]$ or $X = \mathbb{T}$ is a Korovkin system; moreover, $p^+(S) \supset (a, b)$ in the first case, $p^+(S) = \mathbb{T}$ in the second.*

Proof. By a theorem of Krein [11, p. 28] for given $x_0 \in X$, there exists a $g^0 \in G$ that vanishes at x_0 and satisfies $g^0(x) > 0$ for $x \neq x_0$. The only exception is when $X = [a, b]$, $x_0 = a$ (or $x_0 = b$) and m is odd; in this case, the function g^0 could vanish also at the other end point of $[a, b]$. In this exceptional case, there is a function $g^{(1)} \in G$ which vanishes at x_0, and is >0 at the other end point. Then Proposition 13 (with one or two sets A_k) gives the proof at once.

Let X be a compact metric space. About the subspace $G \subset C(X)$ we shall assume now only that it is nonvanishing: For each $x \in X$ there is a $g \in G$ for which $g(x) \neq 0$.

DEFINITION. A point $x_0 \in X$ is a $\mathscr{T}_1{}^+$-quasi-peak-point for G if for each $\epsilon > 0$ and each neighborhood U of x_0, there exists a $\lambda \geqslant 0$ and $g \in G$ with the properties

$$g \leqslant \lambda, \qquad g(x_0) > \lambda - \epsilon,$$
$$g(x) \leqslant \lambda - 1 \qquad \text{if} \quad x \in X \backslash U.$$

We have:

THEOREM 11. *If G is a nonvanishing subset of $C(X)$, then*

$$p_1{}^+(G) \subset q_1{}^+(G) = \partial_1{}^+G, \tag{1.5.8}$$

where $\partial_1{}^+G$ is the $\mathscr{T}_1{}^+$-boundary of G.

The proof is similar to that of Theorem 8; see [18, Theorem 5].

By the definition of boundaries,

$$\partial G \subset \partial_1{}^+G.$$

However, it is easy to see that $\partial G = \partial_1{}^+G$ if G contains a strictly positive function. In this case, there is no difference between the $\mathscr{T}^+$ and the $\mathscr{T}_1{}^+$-quasi-peak-points.

In conclusion, we would like to mention that most of the results of Section 1, for example the sufficiency parts of Theorems 1 and 3, hold for spaces $C(X)$ of continuous functions on an arbitrary compact Hausdorff topological space X. That the necessity parts are not valid, follows from an example of Scheffold [25, p. 7].

2. Finite Korovkin Sets

2.1. *Reformulation of Theorems*

2.1.1. In this chapter, S is a finite subset of $C(X)$,

$$S = \{g_0, g_1, \ldots, g_m\} \tag{2.1.1}$$

of order m; if $g_0 = 1_X$ we write S_0 instead of S.

We assume that the set X contains at least $m + 1$ points and that the g_k are linearily independent. By means of correspondence

$$g = \sum_{k=0}^{m} a_k g_k \leftrightarrow (a_0, a_1, \ldots, a_m)$$

$$l(g) = \sum_{k=0}^{m} a_k l_k \leftrightarrow (l_0, l_1, \ldots, l_m)$$

defined for $g \in G = \operatorname{lin} S$ and $l \in G^*$, we see that G and G^* are both isomorphic to the $m + 1$-dimensional Euclidean space $\mathbb{R}^{m+1}$. Hence, the evaluation map (1.3.1) in Section 1.3:

$$x \to l_x, \qquad l_x(g) = \sum_{k=0}^{m} a_k g_k(x), \qquad (g \in G)$$

can be identified with the map

$$x \to \Phi(x) = (g_0(x), g_1(x), \ldots, g_m(x)) \tag{2.1.2}$$

of X into $\mathbb{R}^{m+1}$, with $X^* = \Phi(X)$. If S is admissible, then Φ is a homeomorphism, and $0 \notin X^*$.

For a set S_0, X^* lies in the hyperplane of $\mathbb{R}^{m+1}$, consisting of points with first coordinate $= 1$. We then replace Φ by

$$x \to \Phi_0(x) = (g_1(x), \ldots, g_m(x)) \in \mathbb{R}^m, \tag{2.1.3}$$

and write $X_0{}^* = \Phi_0(X)$.

In a finitely dimensional space, the convex hull of a closed set is closed. Hence, our definitions of the sets K of Sections 1.3 and 1.4 reduce to:

$$K^+ \text{ is the convex cone in } \mathbb{R}^{m+1} \text{ with the vertex in the origin, generated by } X^*; \tag{2.1.4}$$

$$K_1 = \mathrm{co}(X^* \cup (-X^*)), \qquad K_1^+ = \mathrm{co}(X^* \cup \{0\}), \tag{2.1.5}$$

$$K_0 = \mathrm{co}\, X_0^*. \tag{2.1.6}$$

With $\mathscr{T}$ standing for $\mathscr{T}^+$, $\mathscr{T}_1$, $\mathscr{T}_1^+$, K for the set K^+, K_1, K_1^+ and (B) for the conditions (B^+), (B_1), (B_1^+) of Section 1.4, we have (Šaškin [23, 24]):

THEOREM 1. *A finite subset S of $C(X)$ is a Korovkin set for $\mathscr{T}$ if and only if the corresponding set K satisfies condition (B).*

THEOREM 2. *A finite set S_0 is a Korovkin set for $\mathscr{T}$ if and only if*

(B_0) $\qquad \mathrm{ext}\, K_0 = X_0^*$, *or, equivalently*, $\mathrm{ext}\, K_0 \supset X_0^*$.

The logical relations between these notions are as follows. Each Korovkin set for $\mathscr{T}^+$, or for $\mathscr{T}_1$, is also (trivially) a Korovkin set for $\mathscr{T}_1^+$. There are no other relations; this follows from Examples 1 and 7 of Section 2.2.

2.1.2. A point l_0 of a convex set K in $\mathbb{R}^{m+1}$ is called an *exposed point* of K if there exists a hyperplane H (supporting K) for which $H \cap K = \{l_0\}$. For the set of all exposed points of K we have

$$\exp K \subset \mathrm{ext}\, K. \tag{2.1.7}$$

Similary, a ray ρ of a convex cone K in $\mathbb{R}^{m+1}$ with vertex 0 is an *exposed ray* of K if for some hyperplane H passing through 0, $H \cap K = \rho$. With these notions, we shall interpret the different conditions (P) of Section 1.5.

For a given $x_0 \in X$, condition (P^+) means that there is a point $g_0 = \sum_0^m a_k g_k$ in G for which $l_{x_0}(g_0) = 0$, and $l(g_0) > 0$ for all $l \in X^*$. Then also $l(g_0) = 0$, $l \in \rho_{x_0}$, and $l(g_0) > 0$, for $l \in K^+$. Hyperplanes through 0 in $\mathbb{R}^{m+1} = G^*$ are given by equations $l(g) = 0$ with fixed g. Hence, (P^+) means that the ray ρ_{x_0} is an exposed ray of K^+. Similarly, (P_1) and (P_2) mean that l_{x_0} is an exposed point of K_1 or K_1^+, respectively. Proposition 11 of Section 1.5 becomes: *If ρ_x is an exposed ray (or l_{x_0} an exposed point) of K^+ (or K_1, or K_1^+) with respect to $\mathscr{T}$, then x belongs to the generalized Choquet boundary ∂G of G.*

We need the following result of Straszewicz for subsets of $\mathbb{R}^{m+1}$.

PROPOSITION 1. (i) *If K is a compact convex set in $\mathbb{R}^{m+1}$, then the set* $\exp K$ *is dense in* $\operatorname{ext} K$. (ii) *If K is an acute cone in $\mathbb{R}^{m+1}$, then the set of the exposed rays of K is dense in the set of all extreme rays of K.*

For (i), see [26]; (ii) can be derived easily from (i). As a consequence, we obtain:

THEOREM 3. *For each of the three classes $\mathscr{T}$, the set of the peak points of G is dense in ∂G.*

For example, $p(G)$ is dense in X if S is a Korovkin set.

2.2. *Examples of Korovkin Sets*

The results in this section follow easily from principles developed in Section 2.1.

EXAMPLE 1. Let $X = [a, 1]$, $0 \leqslant a < 1$, $S = \{x, x^2\}$. This set S is not a Korovkin set for $\mathscr{T}^+$. It is a Korovkin set for $\mathscr{T}_1$ if and only if $a \geqslant a_0 = 2^{1/2} - 1$. For $a > a_0$, it is also a strict Korovkin set, with the function g_0 corresponding to x_0, $a \leqslant x_0 \leqslant 1$ given by $g_0(x) = (x/x_0)(2 - x/x_0)$. For $a = a_0$ we have $p_1(G) = (a_0, 1]$. For $\mathscr{T}_1^+$, S is a strict Korovkin set for each $a > 0$.

EXAMPLE 2. Let $f_1, f_2, \ldots, f_r$ be finitely many continuous functions on X, which separate points. Then

$$S_0 = \{1, f_1, \ldots, f_r, f_1^2 + \cdots + f_r^2\} \tag{2.2.1}$$

is a strict Korovkin set for each of the classes $\mathscr{T}$ on X. With the help of the function

$$g_0(x_0) = \sum_{k=1}^{r} (f_k(x) - f_k(x_0))^2 \tag{2.2.2}$$

one sees that each point $x_0 \in X$ is a peak point for $\mathscr{T}$.

EXAMPLE 3. For an arbitrary compact subset X of $\mathbb{R}^r$, the system of functions

$$S_0 = \{1, x_1, \ldots, x_r, x_1^2 + \cdots + x_r^2\}$$

is a strict Korovkin system for each of the classes $\mathscr{T}$.

In Example 2 (and Example 3) one proves that S_0 satisfies the condition (P^+) by finding a function $g(x_0, x)$, which is a linear combination of functions of the system S_0 with coefficients that depend continuously on x_0. This is not always possible:

EXAMPLE 4 [15]. The set $S_0 = \{1, x, x^2 + |x|\}$ is a strict Korovkin set for $X = [-1, +1]$, but there does not exist a function $g(x_0, x)$ of the above type with continuous coefficients. The map Φ_0: $x \to (x, x^2 + |x|)$ transforms X into a strictly convex curve X_0^+ in $\mathbb{R}^2$ that has no continuous selection of supporting straight lines.

One of the most beautiful examples is:

EXAMPLE 5. Let $\mathscr{S}_{r-1}$ be the unit sphere in $\mathbb{R}^r$, given by the equation $x_1^2 + \cdots + x_r^2 = 1$, and let X be a closed subset of $\mathscr{S}_{r-1}$. Then $S_0 = \{1, x_1, \ldots, x_r\}$ is a strict Korovkin system for $\mathscr{T}$. For $\mathscr{T}_1^+$, already the set of the coordinates $S = \{x_1, \ldots, x_r\}$ is a strict Korovkin set, for $\mathscr{T}_1$ this is true precisely when $X \cap (-X) = \varnothing$, that is, when X has no antipodal points.

The following example gives a Korovkin system which is not strict.

EXAMPLE 6 (Šaškin [23]). Let X be the curve formed by the arcs of the circles $x_1^2 + x_2^2 = 4$, $(x_1 \pm 1)^2 + x_2^2 = 1$, as shown on Fig. 1. Then

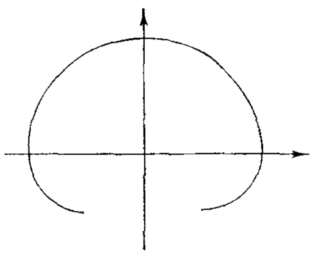

FIGURE 1

the set of functions $S_0 = \{1, x_1, x_2\}$ is a Korovkin set for $\mathscr{T}$, but not a strict Korovkin set. The map Φ_0 transforms X into itself; all points of $K_0 = \operatorname{co} X$ are extreme points of X, but the points $(-1, -1)$ and $(1, -1)$ are not exposed points of K_0.

EXAMPLE 7. Let K^+ be the cone $x_3^2 = x_1^2 + x_2^2$ in $\mathbb{R}^3$. Let Y be any closed bounded star-shaped curve in the x_1, x_2-plane, so that each ray through the origin intersects Y exactly once; let X be the curve on K^+ whose projection is Y. Then $S = \{x_1, x_2, x_3\}$ is a strict $\mathscr{T}^+$ Korovkin set on X, but not a $\mathscr{T}_1$ Korovkin set, if Y is shaped as shown on Fig. 2.

In Example 5, we have seen that there are Korovkin sets for $\mathscr{T}_1^+$, for which $G = \operatorname{lin} S$ does not contain a strictly positive function. For $\mathscr{T}_1$ there is a similar example:

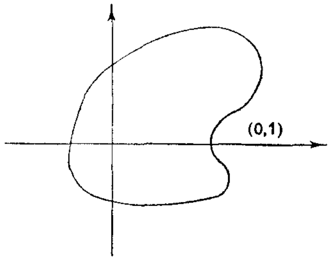

FIGURE 2

EXAMPLE 8. Let X be any closed set in the unit upper semisphere of $\mathbb{R}^3$, which contains only the three points $(-1, 0, 0)$ and $(1/2^{1/2}, \pm 1/2^{1/2}, 0)$ in the equatorial plane $x_3 = 0$. Then $S = \{x_1, x_2, x_3\}$ is a Korovkin set in $C(X)$ for $\mathscr{T}_1$, but if g is any nonnegative function in $G = \operatorname{lin} S$, then g vanishes at each of the three points.

EXAMPLE 9 (M. Wolff). Let G be a dense subspace of $C(X)$; then it can happen that no finite subset in G forms a Korovkin set for $C(X)$, although there are finite Korovkin sets in $C(X)$. Take, for example, G to be the subspace of all piecewise linear functions in $C[0, 1]$.

2.2.2. Let X denote an interval $[a, b]$ or the circle $\mathbb{T}$. A set $S = \{g_0, g_1, ..., g_m\}$ of functions in $C(X)$ is a Čebyšev system on X if each nontrivial linear combination $\sum_{k=0}^{m} a_k g_k$ has at most m distinct zeros on X or, equivalently, if for each set of $m + 1$ distinct points $x_0, ..., x_m$ of X,

$$\det(g_i(x_k)) \neq 0.$$

THEOREM 4. *Each Čebyšev system $S = \{g_0, ..., g_m\}$, $m \geqslant 2$ is a Korovkin system with respect to $\mathscr{T}^+$.*

Proof. According to a theorem of Krein [11, p. 28], for each $x_0 \in X$ there exists a $g \geqslant 0$ that vanishes precisely at the point x_0; the only exception is when $X = [a, b]$, when m is even, and x_0 is one of the end points a, b. Then g may happen to vanish at the other end point too. In the first case, S is a strict Korovkin set. In the exceptional case, we apply Proposition 13 of Section 1.5 with two sets A_1, A_2.

A geometric proof of Theorem 4 bears some interest.

2.3. *The Minimal Order of Korovkin Sets*

The minimal order of a Korovkin set with respect to $\mathscr{T}$, which can exist on a given metric compact X, is a topological invariant $m(\mathscr{T})$. We are able to determine it.

LEMMA 1. *A compact convex body in $\mathbb{R}^n$ is homeomorphic to the unit ball in $\mathbb{R}^n$.*

THEOREM 5 [23, 24]. *Let r_0, r be the minimal dimension of the sphere $\mathscr{S}_{r_0}$ or of the Euclidean space $\mathbb{R}^r$, respectively, into which X can be topologically embedded. Then*

$$m(X, \mathscr{T}^+) = r_0 + 1, \qquad m(X, \mathscr{T}_1) = r, \qquad m(X, \mathscr{T}_1^+) = r_0. \quad (2.3.1)$$

Proof. Case $\mathscr{T}^+$. Let $S = \{g_0, g_1, ..., g_m\}$ be a Korovkin set of minimal order carried by X. Then the functions are linearly independent, and without loss of generality we may assume that $g_0 = 1$.

We map X homeomorphically by means of the map $x \to (g_1(x), ..., g_m(x))$ into the space $\mathbb{R}^m$. If X_0^* is the image of X and $K_0 = \text{co } X_0^*$, then K_0 is a convex compact set with interior points in $\mathbb{R}^m$. Otherwise K_0 would be contained in a hyperplane of $\mathbb{R}^m$, and the functions g_k would not be linearly independent.

Let $y^{(0)}$ be an interior point of K_0, and let $\mathscr{S}_{m-1}$ be the unit sphere in $\mathbb{R}^m$ with center $y^{(0)}$. Theorem 2 shows that X_0^* is situated on the boundary of K_0, and Lemma 1 shows that the central projection from $y^{(0)}$ maps X_0^* homeomorphically into $\mathscr{S}_{m-1}$. Thus, also X is embeddable in $\mathscr{S}_{m-1}$. This proves that $m(X, \mathscr{T}^+) \geqslant r_0 + 1$. The inverse inequality follows from Example 5, Section 2.2: Each subset of $\mathscr{S}_{r_0}$ carries a $\mathscr{T}^+$ Korovkin set of order $r_0 + 1$.

Case $\mathscr{T}_1$. Again let $S = \{g_0, g_1, ..., g_m\}$ be a Korovkin set of minimal order. We consider the homeomorphic embedding Φ of X into $\mathbb{R}^{m+1}$ with image X^*. From the properties of the set K_1 mentioned in Theorem 1, it follows that X^* and $-X^*$ are disjoint and that each ray through the origin meets X^* at most once. The central projection of X^* from the origin onto the unit sphere $\mathscr{S}_m$ with center 0 defines an embedding of X into $\mathscr{S}_m$, which contains no antipodal points. But then X is homeomorphic to a subset of $\mathbb{R}^m$. We obtain $m \geqslant r$. The inverse inequality follows again from Example 5, Section 2.2.

Case $\mathscr{T}_1^+$ can be treated in similar fashion, and is left to the reader.

This proof gives also:

COROLLARY. *If X carries a Korovkin set of order m, then also a strict Korovkin set of the same order.*

EXAMPLE 10. Let $\mathbb{T}_r$, $r = 2, 3, ...$ be the r-dimensional torus. Then

$$m(\mathbb{T}_r, \mathscr{T}^+) = r + 2, \qquad m(\mathbb{T}_r, \mathscr{T}_1) = m(\mathbb{T}_r, \mathscr{T}_1^+) = r + 1.$$

In fact, $\mathbb{T}_r$ is embeddable in $\mathbb{R}^{r+1}$, but not in $\mathscr{S}_r$.

It is not the purpose of this survey to discuss relations of constants r_0, r to other topological invariants of X. We shall mention only a few facts. It follows from Theorem 5 that X has a finite Korovkin set if and only if it has a finite topological dimension $n = \dim X$. According to a theorem of Menger-Nöbeling, one has $n \leqslant r_0 \leqslant r \leqslant 2n + 1$ [11]. It follows from this that

$$n + 1 \leqslant m(X, \mathscr{T}^+) \leqslant 2n + 2 \qquad (2.3.2)$$

$$n \leqslant m(X, \mathscr{T}_1^+) \leqslant m(X, \mathscr{T}_1) \leqslant 2n + 1. \qquad (2.3.3)$$

2.4. *Existence of Linear Relations*

2.4.1. Let X^* be a subset of the Euclidean space $\mathbb{R}^{m+1}$ and let $y \in \mathbb{R}^{m+1}$. The point y is a *linear combination* of points of X^* if there exists a representation

$$y = \sum_{i=1}^{k} a_i y_i, \qquad y_i \in X^*. \qquad (2.4.1)$$

The linear combination (2.4.1) is trivial if one of the coefficients a_i is 1, all other zero; it is *positive* if $a_i \geqslant 0$ for all i; it is *convex* if, in addition, $\sum_{i=1}^{k} a_i = 1$.

According to a theorem of Caratheodory [16, p. 20], a convex combination of points of X^* in $\mathbb{R}^{m+1}$ is also a convex combination of some $m + 2$ points of X^*. According to a theorem of Fenchel [8, p. 9] the number $m + 2$ can be replaced by $m + 1$ if X^* is connected.

The following definition is due to K. Borsuk.

(i) A subset X^* of $\mathbb{R}^{m+1}$ is *k-independent* if no $k + 1$ points of X^* lie in a k-dimensional subspace of $\mathbb{R}^{m+1}$, or, equivalently, if no point of X^* is a nontrivial linear combination of k other points of X^*.

(ii) A subset X^* of $\mathbb{R}^{m+1}$ is k-regular if no $k + 1$ points of X^* lie in a $(k - 1)$-dimensional plane of $\mathbb{R}^{m+1}$; in other words, if no point of X^* is a nontrivial linear combination, with sum of coefficients equal to 1, of some k points of X^+.

These notions can be used in study of systems of functions $S = \{g_0, g_1, ..., g_m\}$ in $C(X)$. We consider the map $\Phi: x \to (g_0(x), ..., g_m(x))$ of the set X into the Euclidean space $\mathbb{R}^{m+1}$. We call the set S (and the map Φ) k-independent if Φ is a homeomorphism and if the image X^* of X under Φ is k-independent. In particular, for a set S_0, let $g_0 = 1$, and let $X_0{}^*$ be the image of X_0 under $\Phi: x \to (g_1(x), ..., g_m(x))$. One sees that S_0 is k-independent if and only if S_0 separates points and if $X_0{}^*$ is k-regular in $\mathbb{R}^m$.

In application to Čebyšev systems we have:

PROPOSITION 2. *Let $S = \{g_0, g_1, ..., g_m\} \subset C(X)$ be a set of functions that separates points of X. Then S is a Čebyšev system on X if and only if it is m-independent.*

Proof. This follows from the criterion for Čebyšev systems by means of determinant (2.2.3) of Section 2.2.

Systems S that are k-independent for some $0 \leqslant k \leqslant m$, are important generalizations of Čebyšev systems, due to Rubinstein. They share some of the properties of the latter. See Žuhovickiĭ [30].

2.4.2. For Korovkin sets, one needs different notions.

(i) A set X^* in $\mathbb{R}^{m+1}$ is k-positively independent if no point of X^* is a nontrivial positive linear combination of some k other points of X^*.

(ii) The set X^* is k-convexly regular if no point of X^* is a nontrivial convex combination of some k points of X.

Using the theorems of Caratheodory and Fenchel, we see that X^* is r-positively independent (convexly regular) for all r if and only if it is $(m+2)$-positively independent (convexly regular). If X^* is connected, we can replace $m + 2$ here by $m + 1$.

One sees easily that 2-positively independent sets are also 2-independent, and that 2-convexly regular sets are even 2-regular.

Conditions (B^+), (B_1), and (B_1^+) of Theorem 1 can be expressed in these terms. We obtain:

THEOREM 6. *An admissible set $S = \{g_0, ..., g_m\}$ in $C(X)$ is*

(i) *a $\mathscr{T}^+$-Korovkin set if and only if 0 does not belong to the convex closure of X^* (or if $G = \operatorname{lin} S$ contains a strictly positive function) and if X^* is $(m+2)$-positively independent.*

(ii) *S is a $\mathscr{T}_1^+$ Korovkin set if and only if X^* is $m + 2$ convexly regular.*

There is also a version for $\mathscr{T}_1$ and for the case when X is connected.

Comparing properties of Čebyšev and Korovkin sets, we obtain:

COROLLARY 1. *If $S = \{g_0, ..., g_m\}$ is a Čebyšev set on a connected set X, then S is a $\mathscr{T}^+$-Korovkin set.* (This is Theorem 4).

By a theorem of Krein [11, p. 28], $G = \operatorname{lin} S$ contains a strictly positive function g_0. Dividing, if necessary, by g_0 we may assume that $g_0 = 1$. Then X_0^* is m-regular (hence m-convexly regular) in $\mathbb{R}^m$; hence, X^* is m-positively independent in $\mathbb{R}^{m+1}$.

EXAMPLE 11. For nonconnected X, this is not necessarily true (Šaškin). Let X be the union of two parabolic arcs $x_2 = -(x_1 + 1)^2$, $-1 \leqslant x_1 \leqslant 0$, $x_2 = (x_1 - 1)^2$, $0 \leqslant x_1 \leqslant 1$ in $\mathbb{R}^2$. Then $S = \{1, x_1, x_2\}$ is not a Korovkin set, but it is a Čebyšev set, because each line in $\mathbb{R}^2$ intersects X at most twice.

COROLLARY 2. *If $S = \{g_0, ..., g_m\}$ is a $\mathscr{T}^+$ Korovkin set on a compact metric space X, then S is 2-independent (for a special case of this, see Volkov* [27]). *In particular, $S = \{g_0, g_1, g_2\}$ is a $\mathscr{T}^+$ Korovkin set on a connected X if and only if S is a Čebyšev system.* (This is Theorem III of Section 0.)

2.5. *Korovkin Sets on Spheres*

It seems to be difficult to characterize, in a simple way, all Korovkin sets of minimal order carried by a given compact metric space X. For the interval and the circle this is done by means of Theorem III of Section 0. In general, problems of this type lead to difficult questions of convex topology in Euclidean spaces. The following result provides an inverse to Corollary 2, Section 2.1. One obtains in this way a complete characterization of minimal Korovkin sets on spheres.

THEOREM 7 (G. G. Lorentz [18]). *A 2-independent set of functions $S = \{g_0, g_1, ..., g_m\}$ on the sphere $\mathscr{S}_{m-1}$ is a strict Korovkin set with respect to $\mathscr{T}^+$ in $C(\mathscr{S}_{m-1})$.*

This theorem has purely geometric formulations. Each 2-independent set is admissible; the map Φ of (1.3.1) is a homeomorphism. The statement of the theorem then translates into: The set of coordinates $\{y_0, ..., y_m\}$ is a strict Korovkin set on $X^* = \Phi(\mathscr{S}_{m-1})$.

As another geometric formulation of Theorem 7 we obtain:

PROPOSITION 3. *Let Y be a topological image of $\mathscr{S}_{m-1}$ in $\mathbb{R}^{m+1}$ that is 2-independent. Then at each point y of Y there exists a strict supporting hyperplane for Y, passing through the origin.*

For a proof, see [18].

Theorem 7 has a special case for a set S_0 which contains the function $g_0 = 1_X$. Using the map Φ_0, we obtain a new geometric interpretation of this special case.

PROPOSITION 4. *Let Y be a topological image of $\mathscr{S}_{m-1}$ in $\mathbb{R}^m$ which is 2-regular. Then at each point of Y there exists a strict supporting hyperplane.*

We leave it to the reader to find a direct proof of this.

In [23], Šaškin has the following assertion: If X is the closure of a bounded region in $\mathbb{R}^m$, then a set $S = \{g_0, ..., g_m\}$ separating points is a $\mathscr{T}^+$-Korovkin set if and only if it is 2-independent. However, the proof given there [23, p. 140] is apparently incorrect, in particular if $m \geqslant 3$.

3. Appendix

The present paper develops the "geometric" (or "Šaškin") theory of Korovkin sets, which is based on the map Φ of the set X into the conjugate space G^*, and the convex properties of the image $\Phi(X)$. Its best applications are for finitely dimensional sets G.

In contrast, a paper by Berens and Lorentz [4] is devoted to the "analytic theory" of Korovkin sets; it is based on the notion of the upper envelope

$$\bar{f}(x) = \inf\{g(x): g \in G, g \geqslant f\}$$

and the lower envelope $\underline{f}(x) = -\overline{(-f)}(x)$. These notions go back to an old paper by Lorentz about almost convergence, but have been used lately by Bauer [2] and Baskakov [3]. This "analytic theory" has a wider scope and is valid for sequences of positive operators T_n from $C(X)$ into a Banach lattice E, with the identity map I replaced by a lattice homomorphism P; it determines not only Korovkin sets but also shadows. However, *contractions* T_n cannot apparently be treated with ease by this theory.

If G is a subspace of $C(X)$, E is a Banach lattice, and P is a given lattice homomorphism of $C(X)$ into E, then $f \in C(X)$ belongs to the *shadow* $\mathscr{S}(G, C(X), E, P)$ of G if for each sequence of positive linear operators T_n from $C(X)$ to E, relations $T_n g \to Pg$, $g \in G$ imply $T_n f \to Pf$. (The set of f where this happens for a fixed sequence T_n is the convergence set of T_n.) The set G is a Korovkin set if its shadow is the whole space $C(X)$. The problem is to describe the shadow of G, if possible.

The basic result of the theory (see [4]) is the following

THEOREM A. *If G contains a strictly positive function g_0, then*

$$\hat{G} \subset \hat{G}_{\operatorname{supp} P} \subset \mathscr{S}(G, C(X), E, P).$$

Here, for any compact $A \subset X$, $\hat{G}_A$ denotes the set of all $f \in C(X)$ for which $\underline{f}(x) = \bar{f}(x)$ for all $x \in A$, and $\hat{G} = \hat{G}_X$.

Unfortunately, in [4] this theorem appears explicitly only under the assumption that $1_X \in G$ and that the functions $g \in G$ separate points, a restriction that is not natural when dealing with shadows. In the form given above, it appears only as a remark in [4, p. 12].

The (generalized Choquet) boundary ∂G of G is the set of all $x \in X$ for which $f(x) = \bar{f}(x)$ for all $f \in C(X)$, or equivalently, of f for which all positive linear extensions of ϵ_x from G to $C(X)$ coincide at f. A point $x \in X$ belongs to ∂G if and only if it is a quasi-peak-point of G (see Lorentz [18]). As a consequence of Theorem A, we have

THEOREM B. *If $\partial G = X$, then G is a Korovkin set.*

Phelps remarks that in the given generality, ∂G does not have properties of a boundary. Whatever its name, the set ∂G is needed here.

About paper [4] we shall remark that Theorem 6 of Section 1 and its proof remain valid under the only assumption that $\| 1_e \| \to 0$ for $\mu e \to 0$, the property of dominated convergence of L^p not being essential. The example of the spaces $M(\alpha, p)$ shows that this is a genuine improvement.

A different procedure, based on lattice-theoretic techniques, was used in [5] to determine the shadow (or the convergence set) for *positive contractions* of L^1 into itself. In [5], this is done when $1_X \in G$; the general case is settled in [6]. Moreover, similar techniques work for fairly general Banach function spaces (spaces with uniformly monotone norm), see [4, Appendix; 6].

We conclude by pointing out three papers dealing with shadows that will soon appear. Priestley [21] discusses shadows of sets S that consist of all pairs g, g^2, $g \in S_1$. The interesting papers of Bernau [7] and Wulbert [29] describe sets in L^p or in a Banach lattice that are shadows of some unspecified set $S \subset E$, with respect to (not necessarily positive) contractions.

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